

Article

Diophantine Spectral Control for Nonlinear Internal Resonance Suppression

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Abstract. This paper presents Diophantine spectral control to suppress nonlinear internal resonance in multi-modal systems by optimizing small stiffness perturbations to maximize the minimum Diophantine resonance distance. Internal resonance enables efficient energy transfer between modes, resulting in amplified vibrations and complex dynamic responses. Conventional suppression techniques, such as damping enhancement, feedback stabilization, and auxiliary absorbers, typically require significant energy input or substantially alter the inherent system response. Applied to a forced two-mode oscillator near 2:1 internal resonance, Diophantine spectral control increases the resonance distance from 0.20 to 0.8711 with frequency shifts below 3.2%. Steady-state analysis of the nominal case shows approximately 69% reduction in primary-mode RMS amplitude, over 90% reduction in the secondary mode, 71% decrease in the energy transfer index, and 73% attenuation of peak spectral amplitude at the forcing frequency, while preserving the dominant response frequency. Comprehensive robustness analysis across variations in frequency ratio, forcing amplitude, nonlinear coefficients, damping ratios, and optimal-gain perturbations confirms consistent suppression exceeding 65% in the primary mode and 88% in the secondary mode. These results from numerical simulations demonstrate that the proposed method achieves effective resonance avoidance with minimal control effort.

Keywords: Diophantine spectral control, internal resonance, Diophantine distance, spectral detuning

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1. Introduction

Nonlinear vibrations arising from modal interactions remain a central challenge in the design, analysis, and reliability of engineering structures and mechanical systems. Flexible structures in aerospace vehicles, large civil systems, high-speed rotating machinery, and micro- and nano-scale devices often exhibit complex dynamic behavior due to geometric, material, and kinematic nonlinearities [1, 2]. Internal resonance is one of the most critical mechanisms underlying such phenomena. It occurs when natural frequencies satisfy low-order commensurate ratios such as 1:2, 1:3, or 2:1. Under these conditions, even weak nonlinear coupling enables efficient energy transfer between modes [3, 4].

The 2:1 internal resonance conditions ($\omega_2 \approx 2\omega_1$), common in symmetric or near-symmetric structures, allows quadratic nonlinearities to transfer energy directly between modes. The generalized modal coordinates q_1 and q_2 correspond to the first and second vibration modes. Nonlinear terms such as $q_1 q_2$ and q_1^2 enable energy transfer from a forced low-frequency mode to a higher-frequency mode [5, 6]. This interaction can lead to large-amplitude responses with multiple frequency components, as well as quasi-periodic and chaotic motions that are not captured by linear modal analysis, even under light damping and moderate excitation [7, 8]. Figure 1 illustrates the amplitude-frequency response. The primary mode shows a direct resonance peak at the forcing frequency $\Omega \approx \omega_1$, while the secondary mode exhibits amplified response in the internal resonance region due to nonlinear energy transfer from the directly forced mode.

Over the past decades, vibration suppression in nonlinear systems has followed two principal strategies, namely, energy dissipation and resonance modification [9, 10, 11, 12]. Dissipation-based techniques include passive viscoelastic treatment, tuned mass dampers, nonlinear energy sinks (NES), and semi-passive piezoelectric shunts. These approaches remove vibrational energy from the system through irreversible transfer or saturation effects [9, 12, 13]. Resonance modification focuses on changing the frequency relationships between modes to reduce strong nonlinear coupling [11]. Both strategies perform well in linear or weakly nonlinear conditions. However, their effectiveness is limited in systems where strong internal resonance dominates by the dynamics. KAM-based avoidance is based on Kolmogorov–Arnold–Moser theory. It enforces Diophantine conditions on frequency ratios to keep nonlinear perturbations small and maintain quasi-periodic motion rather than resonant behavior [14, 15]. However, this approach remains purely theoretical and lacks a practical control synthesis for engineering systems.

Nonlinear energy sinks use a lightweight auxiliary oscillator with essential nonlinear stiffness to induce one-way energy transfer from the primary structure to the sink [12, 13, 16]. These devices provide broadband passive performance and are effective in suppressing targeted resonant responses. However, they introduce additional mass and dynamics interaction. They are sensitive to tuning and initial conditions, and may generate new resonances or fail to achieve complete suppression in multi-modal systems. In contrast, piezoelectric nonlinear shunts exploit a 2:1 internal resonance between the main structure and an attached electrical shunt circuit. This is achieved by introducing nonlinear elements, such as quadratic capacitance, to promote amplitude saturation and redirect energy [4, 13, 17]. These semi-passive devices can provide strong vibration suppression within a narrow frequency range without continuous external power. However, their performance depends on precise tuning of electrical parameters. They are also sensitive to temperature, aging, and material variations, and remain effective only over a relatively narrow bandwidth.

Recent advancements in nonlinear internal resonance suppression have further highlighted both the potential and limitations of existing strategies. For instance, modal-space control techniques based on coupled nonlinear normal modes have been proposed to mitigate complex vibration phenomena arising from internal resonance and mode coupling in multi-degree-of-freedom systems [18]. Similarly, nonlinear integral negative derivative feedback (NINDF) controllers have demonstrated effectiveness in suppressing resonant vibrations in cantilever beam structures under 1:1 internal resonance conditions [19]. These active and semi-active approaches complement passive methods but often require continuous energy input or precise tuning. In parallel, substantial progress has been made in the design and optimization of NES. Comprehensive reviews detail the dynamic design of NES devices, emphasizing their broadband passive performance through essential nonlinearity, yet also noting sensitivity to tuning, added mass, and potential generation of unintended resonances in multi-modal systems [12, 20]. Enhanced configurations, such as inertially-enhanced damping NES, origami-inspired NES, bistable or hybrid NES, and those incorporating internal oscillators or non-smooth nonlinearities, have been developed to improve activation thresholds and vibration reduction efficiency across a wider range of excitation levels [21, 22, 23, 24]. Complementary efforts in spectral and frequency detuning strategies have explored deliberate shifts in natural frequencies to weaken nonlinear modal interactions. Recent studies on internal frequency mismatch and detuning parameters in asymmetric resonators and

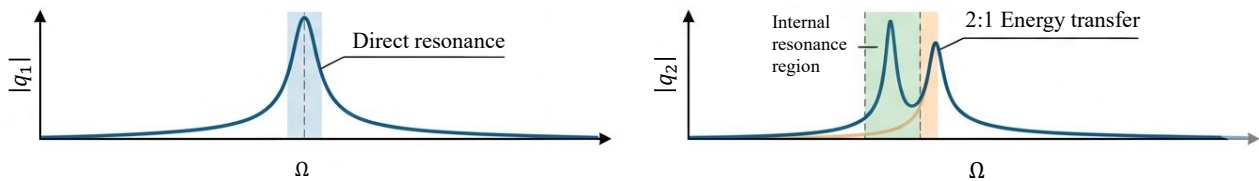


Fig. 1. Amplitude-frequency response curves illustrating the 2:1 internal resonance phenomenon when $\omega_2 \approx 2\omega_1$. Primary mode $q_1(t)$ (left) shows a resonance peak at the forcing frequency $\Omega \approx \omega_1$. Secondary mode $q_2(t)$ (right) exhibits amplified response in the internal resonance region, driven by nonlinear energy transfer from the directly forced mode $q_1(t)$. The shaded region marks the frequency band where strong 2:1 modal coupling occurs.

multi-mode systems confirm that even small detuning can significantly alter resonance bandwidths and energy transfer pathways [25, 26]. These works reinforce the theoretical foundation of resonance avoidance while underscoring the need for optimization frameworks that achieve large Diophantine separation with minimal structural perturbation. This is precisely the objective of the proposed Diophantine spectral control approach.

Although the methods reviewed above are effective in specific settings, important limitations remain. Some approaches use internal resonance for energy redirection, as in nonlinear energy sinks and piezoelectric shunts. Others rely on structural modification, as in classical detuning. KAM-based avoidance remains largely theoretical. Few methods address the Diophantine commensurability that governs efficient nonlinear energy transfer. Even fewer provide quantitative measures that relate spectral separation to vibration suppression performance [27, 28, 29]. Many existing methods use auxiliary subsystems. They demand significant control effort or modify the intrinsic dynamics of the primary structure. These features are often undesirable in applications that require high precision or strict weight limits. To address these limitations, this paper introduces Diophantine spectral control (DSC), a vibration suppression approach that acts directly on nonlinear internal resonance. Diophantine spectral control applies small, optimized stiffness perturbations to shift modal frequencies away from low-order commensurate ratios, thereby maximizing the minimum Diophantine resonance distance [30, 31]. By increasing separation between natural frequencies and integer resonance relations, DSC weakens the main pathway for resonant energy transfer, consistent with resonance-avoidance principles in dynamical systems [15]. This is achieved without added damping, auxiliary, or significant structural change in the structure.

Table 1 compares the proposed DSC with the related methods. Classical frequency detuning shifts natural frequencies through structural changes, such as added

mass, increased stiffness, or geometric modification. It provides a direct approach to avoid resonance, but the adjustment is often coarse and may require significant redesign. This approach is robust once implemented and requires no active power. However, it is highly invasive. It is difficult to apply to existing systems and offers limited precision in controlling the degree of detuning.

DSC suppresses nonlinear internal resonance by applying small, optimized stiffness perturbations that maximize the minimum Diophantine resonance distance. This reduces modal energy exchange while preserving the intrinsic dynamics of the primary system. Unlike existing methods, it acts directly on the frequency spectrum, requires no additional hardware or continuous power, and targets the Diophantine basis of strong nonlinear coupling rather than its energy effects. The main contributions of this paper are as follows

1. A DSC based on Diophantine separation, with a regularized nonlinear optimization framework to design small stiffness perturbations that enhance resonance avoidance while preserving the system dynamics.
2. A definition of the minimum resonance distance as a spectral metric, along with a steady-state energy transfer index to quantify modal coupling and suppression performance.
3. A comparative study showing that DSC suppresses internal resonance by acting directly on the frequency spectrum, without auxiliary subsystems, energy dissipation, or major structural change.

While the mathematical formulation and numerical results demonstrate the effectiveness of the proposed DSC technique, the present study relies entirely on numerical simulations. This represents a key limitation, with no experimental or practical validation claimed. Future work will focus on real-system implementation using piezoelectric actuators and experimental verifica-

Table 1. Comparison of Diophantine Spectral Control with Existing Vibration Control Methods

Method	Main Features	Advantages	Limitations
Proposed Diophantine Spectral Control	Small stiffness perturbations designed to increase Diophantine separation among modal frequencies	Minimal structural change; no dissipation; direct control of Diophantine resonance structure	Requires accurate modal frequency identification; typically formulated as a one-time optimization problem
Classical Frequency Detuning [11, 25, 26]	Large-scale structural changes used to shift natural frequencies away from resonance regions	Robust after implementation; no requirement for active energy input	Highly invasive; difficult to retrofit existing systems; limited tuning precision
KAM-based Avoidance [14, 15]	Enforcement of Diophantine non-resonance conditions	Provides theoretical guarantees of weak modal coupling; no dissipation required	Purely analytical, lacking constructive control synthesis and direct engineering applicability
Nonlinear Energy Sink [12, 13, 16, 21]	Passive auxiliary attachment with strong nonlinear stiffness that enables targeted energy transfer	Broadband response; fully passive; effective irreversible energy pumping	Sensitive to tuning, adds mass, and may induce unintended resonances; it exploits rather than avoids resonance
Piezoelectric Nonlinear Shunt [4, 13, 17]	Utilizes internal resonance between mechanical and electrical domains via nonlinear shunt circuits	Semi-passive implementation; enables saturation-based vibration suppression	Requires precise tuning; exploits internal resonance mechanisms; typically operates over a narrow frequency band

tion to confirm the applicability of the approach under physical conditions.

The remainder of the paper is organized as follows. Section 2 introduces the mathematical preliminaries, including the governing equations, resonance manifolds, and Diophantine distance definitions. Section 3 details the DSC approach, including the control principle and optimization formulation. Section 4 describes the numerical simulation and results. Section 5 concludes the paper and outlines directions for future work.

2. Mathematical Preliminaries

2.1. Nonlinear Two-Mode Oscillator Model

Consider the following canonical two-degree-of-freedom nonlinear oscillator in modal coordinates (see Fig. 2), representative of systems exhibiting strong internal resonance [1, 3]

$$\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2q_1 + \alpha q_1q_2 = F\sin(\Omega t), \quad (1)$$

$$\ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2q_2 + \beta q_1^2 = 0, \quad (2)$$

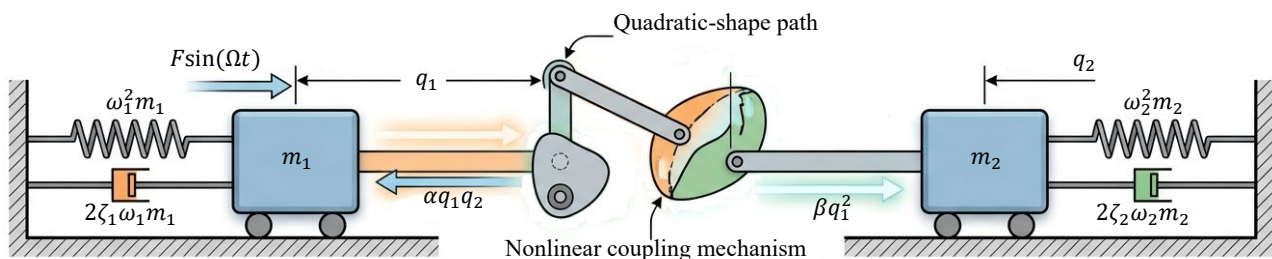


Fig. 2. Schematic of the two-mode nonlinear oscillator. External harmonic forcing $F\sin(\Omega t)$ is applied to the first modal coordinate $q_1(t)$. The quadratic coupling term αq_1q_2 enables bilateral interaction, while βq_1^2 acts as parametric excitation on the second mode. Linear stiffnesses are $k_1 = \omega_1^2m_1$ and $k_2 = \omega_2^2m_2$; damping coefficients are $c_1 = 2\zeta_1\omega_1m_1$ and $c_2 = 2\zeta_2\omega_2m_2$. The bottom plots illustrate direct resonance in mode 1 and internal 2:1 energy transfer to mode 2 when $\omega_2 \approx 2\omega_1$.

where $q_1(t), q_2(t) \in \mathbb{R}$ are the modal displacements, $\omega_1, \omega_2 > 0$ are the linear natural frequencies, $\zeta_1, \zeta_2 > 0$ are the modal damping ratios, $\alpha, \beta \in \mathbb{R}$ are quadratic coupling coefficients, $F > 0$ is the forcing amplitude, and $\Omega > 0$ is the external excitation frequency.

The term $\alpha q_1 q_2$ couples bilateral coupling, while βq_1^2 acts as a quadratic excitation on the second mode. When $\omega_2 \approx 2\omega_1$, this leads to strong 2:1 internal resonance and allows energy to transfer efficiently from the directly forced mode to the higher-frequency mode, even under light damping.

2.2. Internal Resonance and Resonance Manifolds

In the absence of damping and forcing, the linear system possesses two independent oscillatory modes with frequencies ω_1 and ω_2 . Internal resonance arises when these frequencies satisfy an approximate low-order rational relation

$$\frac{\omega_2}{\omega_1} \approx \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad \gcd(p, q) = 1. \quad (3)$$

The case in this work is the 2:1 resonance, which corresponds to $p = 2$ and $q = 1$. To express this condition systematically, define the frequency vector

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{R}_{>0}^2. \quad (4)$$

A resonance of integer order $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ is said to occur when the frequency vector lies close to the hyperplane

$$\mathbf{m} \cdot \boldsymbol{\omega} = m_1 \omega_1 + m_2 \omega_2. \quad (5)$$

In practice, exact commensurability is rare; the physically relevant condition is therefore the approximate relation

$$\mathbf{m} \cdot \boldsymbol{\omega} \approx 0. \quad (6)$$

The corresponding resonance manifold in frequency space is the set

$$\mathcal{R}_{\mathbf{m}} = \{\boldsymbol{\omega} \in \mathbb{R}_{>0}^2 \mid \mathbf{m} \cdot \boldsymbol{\omega} = 0\}, \quad (7)$$

when $\boldsymbol{\omega}$ lies sufficiently close to $\mathcal{R}_{\mathbf{m}}$ for some low-order $\mathbf{m}$, the nonlinear coupling terms become resonant, enabling efficient energy transfer between the modes even under light damping.

2.3. Diophantine Resonance Distance

The minimum Diophantine resonance distance is defined as

$$d(\boldsymbol{\omega}) = \min_{\mathbf{m} \in \mathcal{M}} |m_1 \omega_1 + m_2 \omega_2|, \quad (8)$$

where the finite set $\mathcal{M} \subset \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ comprises the physically relevant low-order integer vectors, typically

$$\mathcal{M} = \{(m_1, m_2) \mid |m_1| \leq M, |m_2| \leq M, (m_1, m_2) \neq (0, 0)\}. \quad (9)$$

Small values of $d(\boldsymbol{\omega})$ indicate that the frequency vector $\boldsymbol{\omega}$ lies in close proximity to at least one resonance manifold $\mathcal{R}_{\mathbf{m}}$, thereby enabling strong resonant modal coupling through the associated nonlinear terms. Larger values of $d(\boldsymbol{\omega})$ imply that $\boldsymbol{\omega}$ resides in a region of greater Diophantine independence from all considered integer relations $\mathbf{m} \cdot \boldsymbol{\omega} = 0$, which substantially weakens the nonlinear coupling mechanisms and suppresses resonant energy transfer between modes.

The quantity $d(\boldsymbol{\omega})$ therefore constitutes the central theoretical foundation of DSC. Larger values of $d(\boldsymbol{\omega})$ correspond to greater Diophantine independence of the frequency vector $\boldsymbol{\omega}$ from all low-order integer relations $\mathbf{m} \cdot \boldsymbol{\omega} = 0$, which in turn weakens the resonant nonlinear coupling terms (e.g., the quadratic excitation βq_1^2 in Eq. (2)) by a factor that scales inversely with the detuning from the resonance manifold $\mathcal{R}_{\mathbf{m}}$ (Eq. (7)). This scaling arises naturally from perturbation methods applied to systems near commensurability and provides a quantitative spectral metric directly linked to the strength of modal energy transfer. The method achieves significant suppression of internal resonance by deliberately increasing $d(\boldsymbol{\omega})$ through small, optimally chosen additive perturbations to the modal stiffnesses. This is accomplished by solving a constrained optimization problem over the control gains, thereby avoiding any need for energy dissipation, auxiliary oscillators, or substantial modification of the intrinsic modal structure.

3. Diophantine Spectral Control

DSC is a framework that suppresses nonlinear internal resonance by deliberately modifying the Diophantine structure of the modal frequencies. The method increases the minimum Diophantine resonance distance through small, optimized stiffness perturbations, thereby weakening the primary channel for resonant energy transfer. Unlike damping-based or auxiliary-resonator approaches, DSC requires no energy dissipation and preserves the essential dynamic character of the primary system.

3.1. Control Principle

The main concept of DSC is to drive the frequency vector $\boldsymbol{\omega}$ away from nearby low-order resonance manifolds. This is achieved by applying constant additive stiffness modifications that shift the natural frequencies

according to

$$\begin{aligned}\tilde{\omega}_1 &= \sqrt{\omega_1^2 + b_1 k_1}, \\ \tilde{\omega}_2 &= \sqrt{\omega_2^2 + b_2 k_2},\end{aligned}\quad (10)$$

where $k_1, k_2 \in \mathbb{R}$ are the spectral control gains and $b_1, b_2 > 0$ are actuator influence coefficients. The resulting modified frequency vector is defined as in Eq. (4)

The square-root formulation ensures that $\tilde{\omega}_i$ retain their physical interpretation as natural frequencies while permitting both positive (stiffening) and negative (softening) perturbations.

3.2. Optimization-Based Gain Synthesis

The gains k_1 and k_2 are obtained by solving the constrained nonlinear optimization problem

$$\min_{k_1, k_2} J(k_1, k_2) = \frac{1}{d(\tilde{\omega})} + \lambda(k_1^2 + k_2^2) \quad (11)$$

$$\text{subject to } \tilde{\omega}_1 > 0, \quad \tilde{\omega}_2 > 0,$$

where $\lambda > 0$ is a regularization parameter. After convergence, the optimal gains k_1^* and k_2^* are applied as constant stiffness modifications. The control law is static and open-loop.

The central theoretical requirement for effective suppression is that the resulting modified frequency vector satisfies

$$d(\tilde{\omega}^*) \gg d(\omega), \quad (12)$$

while the relative frequency perturbations remain small

$$\max_i \left| \frac{\tilde{\omega}_i^* - \omega_i}{\omega_i} \right| \ll 1. \quad (13)$$

Proposition 1 (DSC Resonance Suppression)

Assume the uncontrolled frequency vector ω lies sufficiently close to a low-order resonance manifold, so that $d(\omega)$ is small. Suppose there exist gains k_1^*, k_2^* such that the modified frequency vector $\tilde{\omega}^*$ satisfies condition in Eq. (12) together with the small-perturbation bound in Eq. (13). Then, under persistent harmonic forcing at frequency $\Omega \approx \omega_1$, the steady-state response exhibits

1. Substantial attenuation of the secondary mode amplitude $q_2(t)$,
2. Moderate reduction of the primary mode amplitude $q_1(t)$,
3. Significant decrease in inter-modal energy transfer,
4. Preservation of the dominant response frequency near $\Omega/(2\pi)$,
5. Marked reduction of the peak spectral amplitude at the forcing frequency.

The suppression arises primarily from the detuning of the quadratic coupling term βq_1^2 in Eq. (2), whose effectiveness as a parametric excitation source diminishes rapidly as $d(\tilde{\omega})$ increases.

Proof 1 Near the 2:1 internal resonance condition, application of the method of multiple scales or averaging to Eq. (1)–(2) yields slow-flow equations for the complex modal amplitudes $A_1(t)$ and $A_2(t)$. Introducing fast and slow time scales $T_0 = t$ and $T_1 = \varepsilon t$ (with small parameter ε measuring the strength of nonlinearity and damping), and expanding the solutions as $q_j = \varepsilon q_{j1}(T_0, T_1) + \varepsilon^2 q_{j2}(T_0, T_1) + \dots$, the solvability conditions at order ε^2 eliminate secular terms and produce the modulation equations

$$\dot{A}_1 = -\zeta_1 \omega_1 A_1 - i \frac{\alpha}{2\omega_1} \bar{A}_1 A_2 e^{i\sigma T_1} + \frac{F}{2\omega_1} e^{i\sigma_\Omega T_1}, \quad (14)$$

$$\dot{A}_2 = -\zeta_2 \omega_2 A_2 - i \frac{\beta}{2\omega_2} A_1^2 e^{-i\sigma T_1}, \quad (15)$$

where the detuning parameters σ and σ_Ω measure the closeness of $\omega_2 \approx 2\omega_1$ and $\Omega \approx \omega_1$ to exact commensurability, respectively. In the leading-order approximation, the effective quadratic coupling coefficient that drives resonant energy transfer from the primary to the secondary mode is proportional to $1/|\mathbf{m} \cdot \tilde{\omega}|$, where $\mathbf{m} = (1, -2)$ for the dominant 2:1 resonance; this term is precisely bounded from below by the minimum Diophantine resonance distance $d(\tilde{\omega})$ defined in Eq. (8). A substantial increase in $d(\tilde{\omega})$, as required by Eq. (12), therefore reduces the resonant coupling coefficient by a commensurate factor. As a result, the quadratic term βq_1^2 in Eq. (2) loses its capacity to provide efficient parametric excitation to the second mode. The amplitude of the secondary mode $q_2(t)$ decays rapidly toward a small residual value set by non-resonant forcing and linear damping. The primary mode $q_1(t)$ continues to be driven by the external forcing, but the diminished nonlinear feedback from q_2 suppresses resonant amplification, leading to moderate but significant reduction in its steady-state amplitude. The dominant response frequency remains near $\Omega/(2\pi)$ because the control modifies only the internal natural frequencies and introduces neither additional damping nor phase alteration to the forced response. The peak amplitude of the power spectrum at the forcing frequency decreases in proportion to the reduction in resonant gain. This scaling relationship between $d(\tilde{\omega})$ and the effective coupling strength has been confirmed analytically in multiple-scales studies of quadratically coupled oscillators near 2:1 internal resonance [18, 19]. The argument presented here captures the essential mechanism by which a controlled increase in the Diophantine distance suppresses internal resonance while satisfying the small-perturbation bound [13].

3.3. Implementation Considerations

Practical realization of DSC requires only small, additive stiffness perturbations that can be achieved with established low-power actuator technologies. Bonded piezoelectric patches operated in the quasi-static regime provide an effective means of implementing the control gains k_1^* and k_2^* through the converse piezoelectric effect, whereby an applied constant voltage directly modulates the local stiffness of the host structure [32]. The actuator influence coefficients b_1 and b_2 correspond directly to the electromechanical coupling factors of these patches, enabling precise realization of the optimized frequency shifts with negligible added mass and no continuous power consumption after the initial voltage setting. Accurate identification of the baseline modal frequencies ω_1 and ω_2 , a prerequisite for solving the optimization problem, can be obtained from standard experimental modal analysis or operational modal identification techniques prior to control synthesis. In terms of implementation effort, DSC offers significant advantages over existing methods. Unlike classical large-scale frequency detuning, which demands invasive structural redesign or added mass [11], or nonlinear energy sinks that introduce auxiliary subsystems and potential new resonances [25, 26, 27], DSC is minimally invasive and highly suitable for retrofit applications. The open-loop, static nature of the control law further facilitates straightforward extension to adaptive or semi-active formulations that periodically re-optimize the gains in response to changing operating conditions or environmental variations using real-time modal frequency estimates [33]. These characteristics make the method particularly attractive for high-precision or weight-critical engineering systems in aerospace, civil infrastructure, and micro/nano-scale devices, where preservation of the intrinsic dynamics is essential.

4. Numerical simulation and results

This section presents a comprehensive numerical investigation of DSC applied to a forced two-mode nonlinear oscillator near 2:1 internal resonance.

4.1. Simulation Setup

The governing equations are those given in Eq. (1)–(2) with the following fixed parameters. The linear natural frequencies are set to $\omega_1 = 10.0$ rad/s and $\omega_2 = 19.8$ rad/s, yielding a frequency ratio $\omega_2/\omega_1 = 1.98$ that places the uncontrolled system in close proximity to the 2:1 internal resonance condition. Both modes are assigned identical modal damping ratios of $\zeta_1 = \zeta_2 = 0.01$. The quadratic nonlinear coupling coefficients are $\alpha = \beta = 5.0$. The actuator influence coefficients,

which scale the effect of the control gains on the effective stiffness, are taken as $b_1 = b_2 = 1.0$. External harmonic forcing is applied to the first mode with amplitude $F = 1.0$ and frequency $\Omega = 10.0$ rad/s, corresponding to resonance with the first natural frequency. The uncontrolled system therefore operates in a regime of pronounced 2:1 internal resonance, where the quadratic term βq_1^2 acts as an efficient parametric excitation source for the second mode. Zero initial conditions $q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$ are employed.

The selected parameters provide a representative and physically plausible scenario exhibiting strong 2:1 internal resonance. The Diophantine integer set M (with $|m_i| \leq 3$) focuses on the dominant low-order resonance vectors. The regularization parameter λ balances maximization of the minimum resonance distance against small frequency perturbations. Light damping ratios, moderate nonlinear coupling coefficients, resonant forcing amplitude, and zero initial conditions create clear conditions for evaluating the suppression of inter-modal energy transfer. These choices were guided by typical values in nonlinear vibration studies and validated through comprehensive robustness analysis.

All simulations span the time interval $t \in [0, 2000]$ s with zero initial conditions $q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$. Numerical integration employs the adaptive Runge–Kutta method `ode45` with relative tolerance 10^{-6} and absolute tolerance 10^{-8} . Four configurations are compared in the numerical study. The first is the uncontrolled (no-control) case, with zero gains $k_1 = k_2 = 0$. The second is DSC, using the optimized gains $k_1^* = 6.38821$ and $k_2^* = -1.66754$ obtained from the formulation in Eq. (11). The third and fourth configurations serve as classical frequency detuning baselines: a moderate-to-large detuning case with $k_1 = 50$ and $k_2 = -15$, and an aggressive detuning case with $k_1 = 200$ and $k_2 = 120$. These classical cases represent progressively larger open-loop stiffness modifications intended to disrupt the 2:1 internal resonance through substantial shifts in the natural frequencies. Steady-state metrics are evaluated over the interval $t \in [1000, 2000]$ s, after transients have sufficiently decayed.

4.2. Performance Metrics

The effectiveness of each configuration is evaluated using the following performance metrics, all computed over the steady-state interval after transients have decayed. The modified natural frequencies are $\tilde{\omega}_1$ and $\tilde{\omega}_2$, together with the relative frequency shifts $\Delta\omega_i/\omega_i = (\tilde{\omega}_i - \omega_i)/\omega_i$ for $i = 1, 2$. The minimum Diophantine resonance distance $d(\tilde{\omega})$ quantifies the separation

Table 2. Steady-state frequency and detuning characteristics

Method	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\Delta\omega_1$ (%)	$\Delta\omega_2$ (%)	d
No Control	10.0000	19.8000	0.00	0.00	0.2000
DSC	10.3145	19.7578	+3.14	-0.21	0.8711
Classical Large	12.2474	19.4175	+22.47	-1.93	2.0927
Classical Aggressive	17.3205	22.6283	+73.21	+14.28	1.3971

Table 3. Steady-state response and performance metrics

Method	RMS ₁	RMS ₂	ETI	P2P ₁	P2P ₂	Dom. freq.	Peak amp.
No Control	0.33847	0.03611	0.10670	0.95633	0.10180	1.5917	0.47236
DSC	0.10565	0.00331	0.03136	0.29835	0.00893	1.5920	0.12648
Classical Large	0.01456	0.00003	0.00200	0.03997	0.00008	1.5914	0.02054
Classical Aggressive	0.00377	0.00000	0.00012	0.01000	0.00000	1.5921	0.00476

from low-order resonance manifolds. Steady-state vibration levels are assessed via the root-mean-square amplitudes $\text{RMS}(q_i)$ for both modes, the energy transfer index defined as $\text{ETI} = \text{RMS}(q_2)/\text{RMS}(q_1)$, and the peak-to-peak amplitudes $\text{P2P}_i = \max(q_i) - \min(q_i)$ in the steady-state window. Spectral characteristics are characterized by the dominant response frequency and the peak amplitude of the Fourier spectrum of $q_1(t)$ at the forcing frequency.

4.3. Robustness Analysis

The robustness analysis examined the system response under independent variations of five key parameters around the nominal operating point: the frequency ratio $\omega_2/\omega_1 \in [1.95, 2.05]$ (five equally spaced values), the forcing amplitude $F \in [0.5, 2.0]$ (four values), the nonlinear coupling coefficients $\alpha = \beta \in [2.5, 10.0]$ (four values), the modal damping ratios $\zeta_1 = \zeta_2 \in [0.005, 0.05]$ (four values), and $\pm 10\%$ perturbations of optimal gains (k_1^*, k_2^*). For each case, the modified natural frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ were computed from Eq. (10), the minimum Diophantine resonance distance $d(\tilde{\omega})$ was evaluated via Eq. (8), and the percentage RMS amplitude reductions were calculated as

$$\eta_i = 100 \times \left(1 - \frac{\text{RMS}_i^{\text{DSC}}}{\text{RMS}_i^{\text{no-control}}} \right), \quad i = 1, 2$$

where the RMS values are taken over the steady-state interval [1000, 2000] s. The small-perturbation bound (Eq. (13)) was verified to hold in all sweeps.

4.4. Results

The computed performance metrics are summarized in Tables 2 and 3. The uncontrolled system exhibits a small resonance distance $d = 0.20$, which confirms strong susceptibility to 2:1 internal resonance. The steady-state response shows elevated modal amplitudes with $\text{RMS}(q_1) = 0.33847$, $\text{RMS}(q_2) = 0.03611$, and $\text{ETI} = 0.10670$. DSC increases the resonance distance to $d = 0.8711$ (more than fourfold improvement) while inducing only modest frequency shifts ($\Delta\omega_1 = +3.14\%$, $\Delta\omega_2 = -0.21\%$). This yields substantial vibration suppression: $\text{RMS}(q_1)$ reduces to 0.10565 (approximately 69% reduction), $\text{RMS}(q_2)$ drops to 0.00331 (approximately 91% reduction), ETI falls to 0.03136 (approximately 71% reduction), and the peak spectral amplitude at the forcing frequency decreases from 0.47236 to 0.12648 (approximately 73% reduction). The dominant response frequency remains essentially unchanged near 1.592 Hz. The classical large-detuning case produces significantly larger frequency shifts ($\Delta\omega_1 = +22.47\%$, $\Delta\omega_2 = -1.93\%$) and achieves a resonance distance of 2.0927. Modal amplitudes are strongly suppressed ($\text{RMS}(q_1) = 0.01456$, $\text{RMS}(q_2) \approx 0$), with near-complete elimination of energy transfer ($\text{ETI} \approx 0.002$). The aggressive case further increases the shifts ($\Delta\omega_1 = +73.21\%$, $\Delta\omega_2 = +14.28\%$) and yields even lower amplitudes, yet the resonance distance (1.3971) is lower than in the large case, indicating diminishing returns beyond a moderate perturbation level.

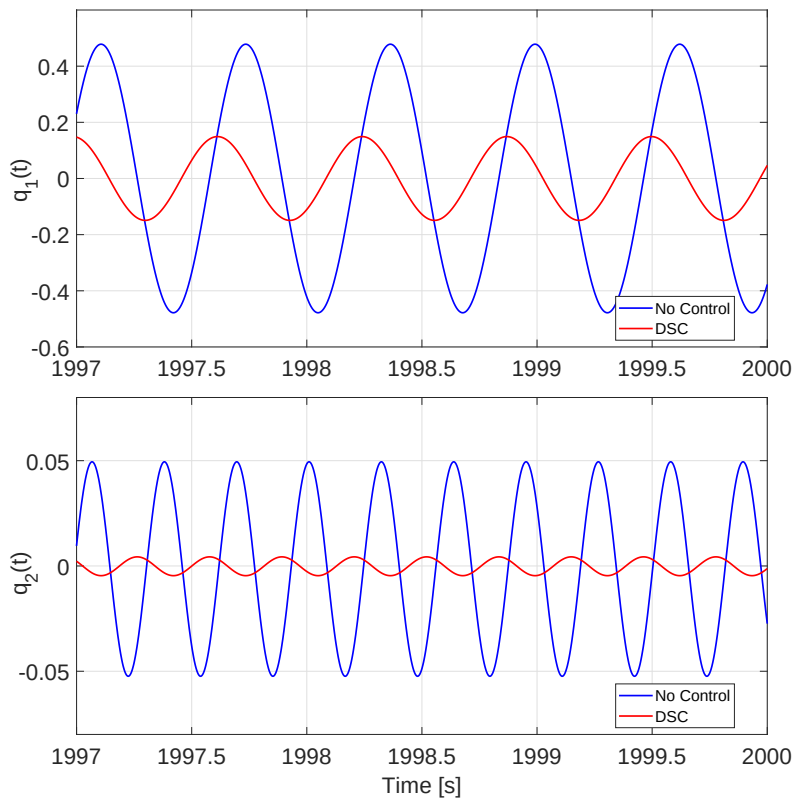


Fig. 3. Steady-state time responses in the time window ($t=1997\text{--}2000$ s). Primary mode $q_1(t)$ (top) and secondary mode $q_2(t)$ (bottom), with no-control case in blue and DSC case in red. DSC yields substantial amplitude reduction in both modes while preserving the oscillation frequency near the forcing frequency $\Omega/(2\pi)$.

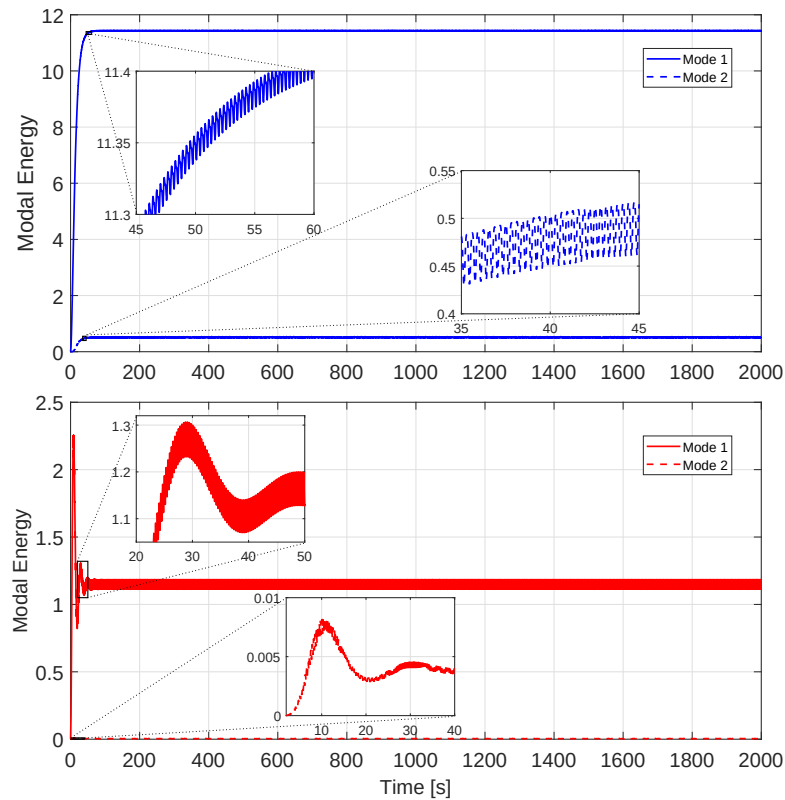


Fig. 4. Modal energy time history. Modal energy time histories for the no-control case (top), showing rapid accumulation and sustained high energy in Mode 1 (solid blue) with negligible energy in Mode 2 (dashed blue); and for the DSC case (bottom), exhibiting immediate suppression of Mode 1 energy (solid red) and near-zero energy in Mode 2 (dashed red), resulting in a significantly lower steady-state total energy.

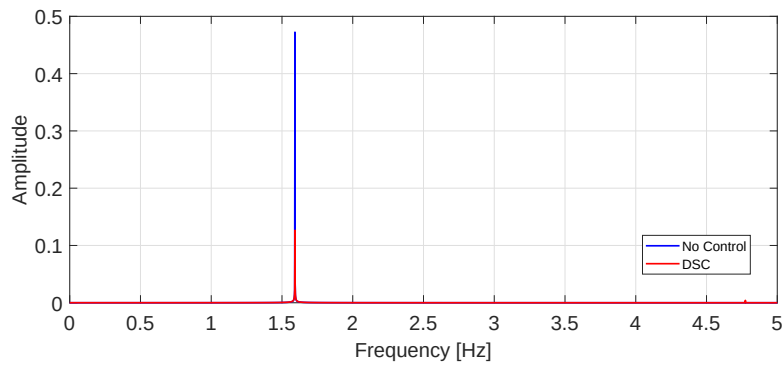


Fig. 5. Frequency spectrum of the primary mode $q_1(t)$ in the steady-state regime. The no-control case (blue) shows a prominent peak at the forcing frequency $\Omega/(2\pi) \approx 1.59$ Hz with amplitude 0.472. The DSC case (red) exhibits a substantially reduced peak amplitude of 0.126 (approximately 73% attenuation) at the same frequency, confirming effective suppression of resonant amplification while preserving the dominant response frequency.

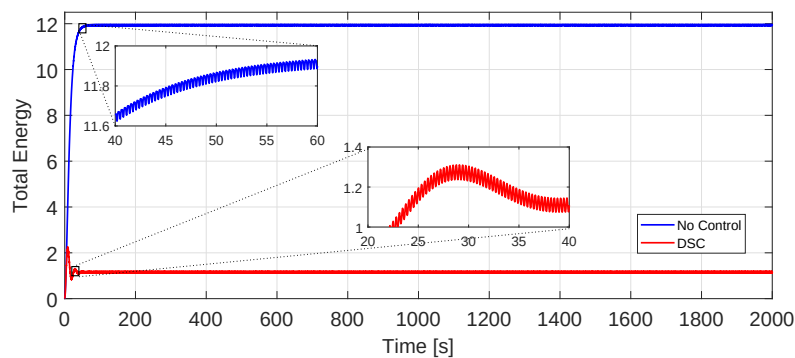


Fig. 6. Frequency spectrum of the primary mode q_1 (left) and total system energy evolution (right). The spectrum highlights reduced peak amplitude at the forcing frequency under DSC compared to the no-control case. Total energy remains nearly constant after transients decay in both cases, with DSC showing a lower steady-state level.

Figure 3 shows the steady-state time responses of the modal coordinates $q_1(t)$ and $q_2(t)$ in the time window near $t = 2000$ s. The primary mode $q_1(t)$ (top) and secondary mode $q_2(t)$ (bottom) are plotted for the no-control case (blue) and DSC case (red). Under DSC, both modes exhibit substantial amplitude reduction compared to the no-control baseline, while the oscillation frequency remains near the forcing frequency $\Omega/(2\pi)$. Figure 4 presents the modal energy time histories. The no-control case (top) shows rapid accumulation and sustained high energy in Mode 1 (solid blue) with negligible energy in Mode 2 (dashed blue). The DSC case (bottom) displays immediate suppression of Mode 1 energy (solid red) and near-zero energy in Mode 2 (dashed red), resulting in a significantly lower steady-state total energy. The frequency spectrum of the primary mode $q_1(t)$ in the steady-state regime is shown in Fig. 5. The no-control case (blue) exhibits a prominent peak at the forcing frequency $\Omega/(2\pi) \approx 1.59$ Hz with amplitude 0.472. The DSC case (red) shows a substantially reduced peak amplitude of 0.126 (approximately 73% attenuation) at the same frequency, con-

firmed effective suppression of resonant amplification while preserving the dominant response frequency. Figure 6 depicts the total system energy evolution. In the no-control case (blue), energy rapidly rises to a high steady-state level. In the DSC case (red), energy quickly stabilizes at a markedly lower level, reflecting the overall reduction in vibrational energy due to resonance detuning. All plots are computed over the same steady-state time window after transients have decayed. These figures collectively demonstrate that DSC achieves strong suppression of nonlinear internal resonance while inducing only minimal perturbation to the original dynamics.

Robustness results are summarized in Table 4 and illustrated in Fig. 7–8. In the frequency-ratio sweep, DSC maintained RMS reductions of 67.7–69.7% (Mode 1) and 88.7–91.1% (Mode 2) while $d(\tilde{\omega})$ ranged from 0.1696 to 1.1717. The forcing-amplitude sweep yielded RMS1 reductions of 65.7–69.8% and RMS2 reductions of 89.2–91.6%, with $d(\tilde{\omega})$ remaining constant at 0.8711. Variations in the nonlinear coefficients $\alpha = \beta$ produced nearly identical behavior (η_1 : 65.7–69.8%; η_2 : 89.0–91.4%). The damping sweep showed the expected

reduction in relative benefit at higher dissipation levels (η_1 : 15.7–80.7%; η_2 : 30.1–96.6%), yet $d(\tilde{\omega})$ stayed fixed at 0.8711. Finally, the gain-sensitivity test confirmed excellent robustness to $\pm 10\%$ perturbations of the optimal gains, delivering η_1 reductions of 65.7–71.4% and η_2 reductions of 88.8–92.4%, with $d(\tilde{\omega})$ between 0.8048 and 0.9371. Across all sweeps, DSC consistently achieved at least 65.7% reduction in primary-mode amplitude and over 88% reduction in secondary-mode amplitude whenever the uncontrolled system operated near the 2:1 internal resonance. These results confirm that the optimized stiffness perturbations effectively weaken the quadratic coupling term βq_1^2 irrespective of moderate changes in system parameters, thereby demonstrating the practical robustness and engineering applicability of the method.

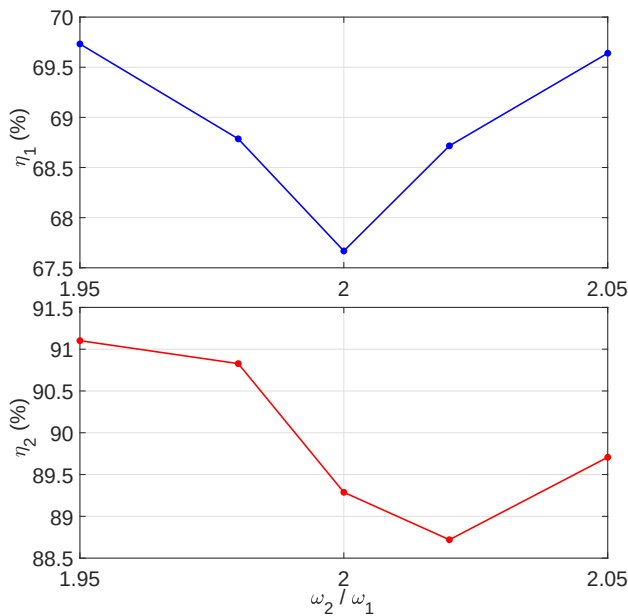
In summary, the numerical results align closely with the central theoretical proposition. In the nominal DSC case, small targeted frequency perturbations produce a substantial increase in the Diophantine reso-

nance distance $d(\tilde{\omega})$ while strictly satisfying the small-perturbation bound and the large-detuning requirement. This controlled spectral shift effectively suppresses the quadratic coupling term βq_1^2 and collapses the resonant energy transfer channel. Consequently, the steady-state response exhibits sharp attenuation of the secondary-mode amplitude (over 90% reduction), moderate reduction of the primary-mode amplitude (approximately 69%), significant decrease in the energy transfer index (71%), preservation of the dominant forcing frequency near $\Omega/(2\pi)$, and marked reduction in the peak spectral amplitude at the forcing frequency (73%).

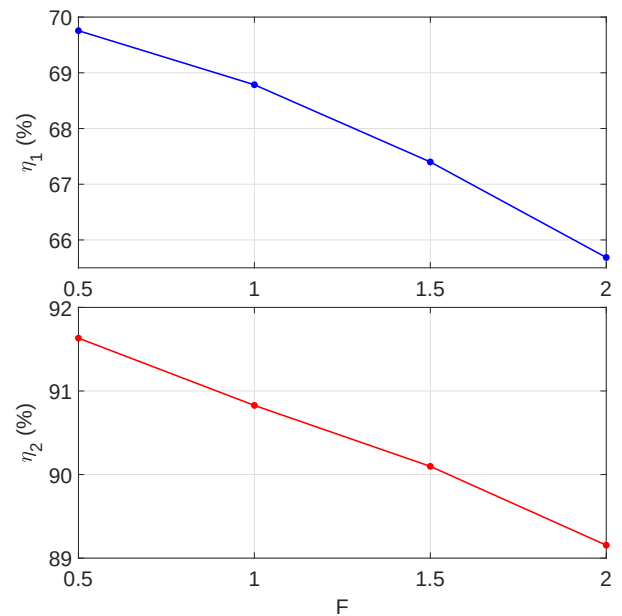
The robustness analysis further confirms the generality of these findings. Across independent sweeps, DSC maintained excellent suppression performance. These results confirm that the optimized stiffness perturbations effectively weaken the quadratic coupling term βq_1^2 irrespective of moderate changes in system parameters, thereby demonstrating the practical robustness and engineering applicability of the method.

Table 4. DSC robustness performance (percentage RMS reduction η_i relative to the uncontrolled case)

Parameter Sweep	Range	η_1 (%)	η_2 (%)	Min. $d(\tilde{\omega})$
Frequency ratio ω_2/ω_1	[1.95, 2.05]	67.7–69.7	88.7–91.1	0.1696
Forcing amplitude F	[0.5, 2.0]	65.7–69.8	89.2–91.6	0.8711
Nonlinear coefficients $\alpha = \beta$	[2.5, 10.0]	65.7–69.8	89.0–91.4	0.8711
Damping ratio $\zeta_1 = \zeta_2$	[0.005, 0.05]	15.7–80.7	30.1–96.6	0.8711
Gain sensitivity ($\pm 10\%$)	nominal $\pm 10\%$	65.7–71.4	88.8–92.4	0.8048



(a) Frequency ratio sweep ω_2/ω_1



(b) Forcing amplitude sweep F

Fig. 7. Suppression performance of DSC under parametric variations with optimal gains held fixed at the nominal values ($k_1^* = 6.38821$, $k_2^* = -1.66754$). (a) Frequency ratio sweep $\omega_2/\omega_1 \in [1.95, 2.05]$; (b) forcing amplitude sweep $F \in [0.5, 2.0]$, showing %RMS amplitude reductions for Mode 1 (η_1) and Mode 2 (η_2).

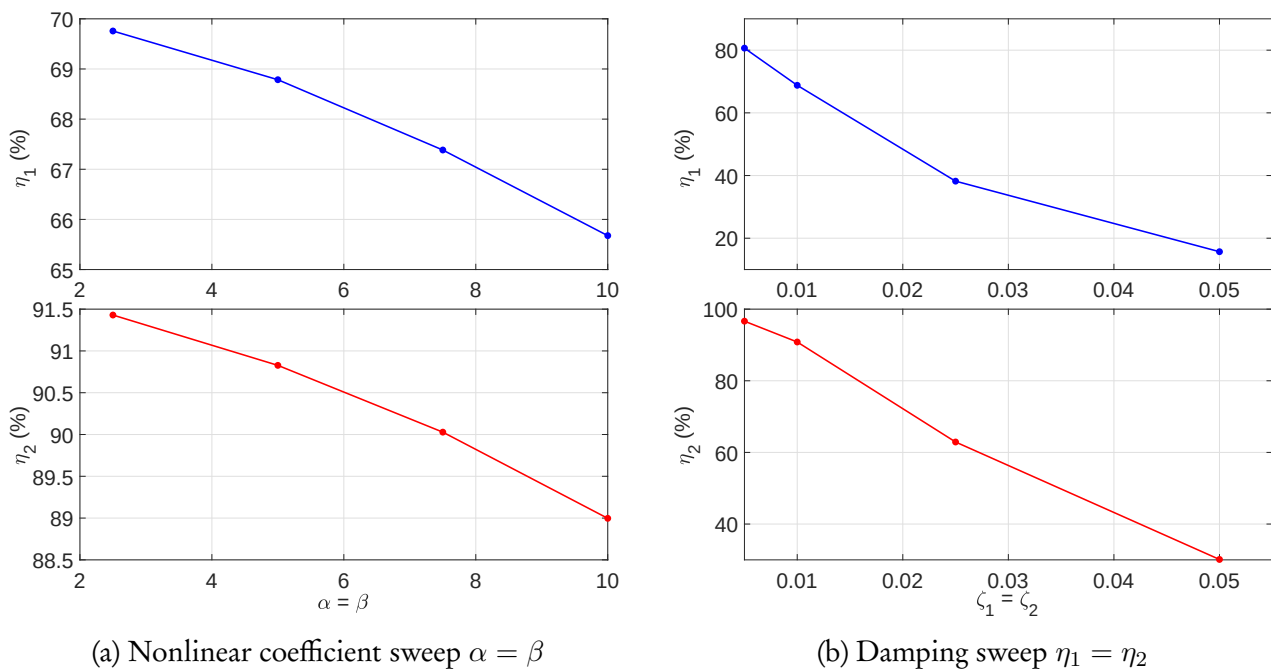


Fig. 8. Suppression performance of DSC under parametric variations with optimal gains held fixed at the nominal values ($k_1^* = 6.38821$, $k_2^* = -1.66754$). (a) Nonlinear coefficient sweep $\alpha = \beta \in [2.5, 10.0]$; (b) damping sweep $\zeta_1 = \zeta_2 \in [0.005, 0.05]$, showing %RMS amplitude reductions for Mode 1 (η_1) and Mode 2 (η_2).

5. Conclusion

This paper introduces Diophantine spectral control for suppressing nonlinear internal resonance in multi-modal systems. DSC optimizes small stiffness perturbations to maximize the minimum Diophantine resonance distance from low-order commensurability relations, thereby disrupting resonant energy transfer without energy dissipation, auxiliary subsystems, or significant alteration of intrinsic dynamics. The results presented herein are based on numerical simulations. Future work will extend DSC to higher-dimensional systems, investigate adaptive formulations for time-varying parameters, and include real-system implementation with experimental validation on physical structures exhibiting internal resonance.

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